Various Identities (degree 2, 3 & 4)

Degree Two:

$$a^2 + 3b^2 = c^2 + 3d^2$$

Solution (a,b,c,d)=[(x-13y),(5x-9y),(7x-7y),(3x-11y)]

For (x,y)=(3,1), we get: $5^2 + 3(3)^2 = 7^2 + 3(1)^2$

Degree Three

$$(13x^2 + 22xy + 13y^2)^3$$
$$= (7x^2 - 14xy - 41y^2)^3 + (6x^2 + 12xy - 18y^2)^3$$
$$+(x^2 - 2xy + 49y^2)^3 + (5x^2 + 38xy + 5y^2)^3$$
$$+ (8x^2 + 16xy - 24y^2)^3$$
$$+ (10x^2 + 20xy - 30y^2)^3$$

For (x,y)=(2,1), we get: $109^3 = (101,50,49,30,40,-41)^3$

Degree Four

$$(a + b + c)^4 + a^4 + b^4 + c^4 + 3(d)^4$$
$$= (a + b)^4 + (b + c)^4 + (c + a)^4$$

$$Where\ (d)^4 = -4(abc)(a + b + c)$$

For (a,b,c,d)=(2,-2,1,2), we get: $(2,2,1,1)^4 + 3(2)^4 = (1)^4 + (3)^4$

New solution from (known) solutions of equations

Degree two:

$$a^2+b^2=c^2+d^2$$

Known solution is: (a,b,c,d)=(8,1,4,7)

Hence we parameterize at (a,b,c,d)=((8+t),(1+kt),(4+t),(7+kt))

$$We\ get\ (a,b,c,d) = \big((3x+24y),(2x+3y),(3x+12y),(2x+21y)\big)$$

For (x,y)=(3,1) we get: $(3)^2+(11)^2=(7)^2+(9)^2$

Degree three:

We have Identity shown below:

$$(a+b+c)^3+a^3+b^3+c^3 = (a+b)^3+(b+c)^3+(c+a)^3+(d)^3$$

$$Where: \quad d^3=(6abc)$$

Since known solution is: (a,b,c,d)=(4,9,1,6) we parametrize as,

(a,b,c)=[(4+t),(9+t),(1+kt)] & we get below stated solution:

a=$(324x^2-72xy+4y^2)$

b=$(144x^2+72xy+9y^2)$

c= $(36x^2+5xy-y^2)$

For (x,y)=(1,1) we get: (a,b,c,d)=(256,225,-40,-240)

Hence solution is: $(481,185,216,40)^3 = (256,225,441,240)^3$

Degree four:

We have Identity shown below:

$$(a+b+c)^4 + a^4 + b^4 + c^4 + 3(d)^4$$
$$= (a+b)^4 + (b+c)^4 + (c+a)^4$$

$$Where: \quad d^4 = -4abc(a+b+c)$$

Since known solution is: (a,b,c,d)=(2-2,1) we parametrize as:

(a,b,c)=[(t+2),(t-2),(kt+1)]

The resulting cubic equation has solution at (k,t)=(1,-13/7)

Hence we get (a,b,c)=(1,-27,-6) and so the new solution is:

For (a,b,c,d)=(1,-27,-6,12), we get:

$$[(32,1,27,6)^4 + 3(12)^4] = [26,33,5]^4$$
